GUIDE 98-3/Suppl.1

Uncertainty of measurement

Part 3:
Guide to the expression of uncertainty in measurement (GUM:1995)

Supplement 1:
Propagation of distributions using a Monte Carlo method

First edition 2008

ISO copyright office
Case postale 56 • CH-1211 Geneva 20
Tel. + 41 22 749 01 11
Fax + 41 22 749 09 47
E-mail copyright@iso.org
Web www.iso.org

Published in Switzerland

Contents

Page

Foreword

ISO (the International Organization for Standardization) is a worldwide federation of national standards bodies (ISO member bodies). The work of preparing International Standards is normally carried out through ISO technical committees. Each member body interested in a subject for which a technical committee has been established has the right to be represented on that committee. International organizations, governmental and non-governmental, in liaison with ISO, also take part in the work. ISO collaborates closely with the International Electrotechnical Commission (IEC) on all matters of electrotechnical standardization.

International Standards are drafted in accordance with the rules given in the ISO/IEC Directives, Part 2.

Draft Guides adopted by the responsible Committee or Group are circulated to the member bodies for voting. Publication as a Guide requires approval by at least 75 % of the member bodies casting a vote.

Attention is drawn to the possibility that some of the elements of this document may be the subject of patent rights. ISO shall not be held responsible for identifying any or all such patent rights.

This first edition of Supplement 1 to ISO/IEC Guide 98-3 has been prepared by Working Group 1 of the JCGM, and has benefited from detailed reviews undertaken by member organizations of the JCGM and National Metrology Institutes. For further information, see the Introduction (0.2).

ISO/IEC Guide 98 consists of the following parts, under the general title *Uncertainty of measurement*:

— *Part 1: Introduction to the expression of uncertainty in measurement*

— *Part 3: Guide to the expression of uncertainty in measurement (GUM:1995)*

The following parts are planned:

— *Part 2: Concepts and basic principles*

— *Part 4: Role of measurement uncertainty in conformity assessment*

— *Part 5: Applications of the least-squares method*

ISO/IEC Guide 98-3 has one supplement.

— *Supplement 1: Propagation of distributions using a Monte Carlo method*

The following supplements to ISO/IEC Guide 98-3 are planned:

— *Supplement 2: Models with any number of output quantities*

— *Supplement 3: Modelling*

Note that in this document, GUM is used to refer to the industry-recognized publication, adopted as ISO/IEC Guide 98-3:2008. When a specific clause or subclause number is cited, the reference is to ISO/IEC Guide 98-3:2008.

Introduction

0.1 General

This Supplement to the *Guide to the expression of uncertainty in measurement (GUM)* is concerned with the propagation of probability distributions through a mathematical model of measurement [ISO/IEC Guide 98-3:2008, 3.1.6] as a basis for the evaluation of uncertainty of measurement, and its implementation by a Monte Carlo method. The treatment applies to a model having any number of input quantities, and a single output quantity.

The described Monte Carlo method is a practical alternative to the GUM uncertainty framework [ISO/IEC Guide 98-3:2008, 3.4.8]. It has value when

a) linearization of the model provides an inadequate representation or

b) the probability density function (PDF) for the output quantity departs appreciably from a Gaussian distribution or a scaled and shifted t-distribution, e.g. due to marked asymmetry.

In case a), the estimate of the output quantity and the associated standard uncertainty provided by the GUM uncertainty framework might be unreliable. In case b), unrealistic coverage intervals (a generalization of "expanded uncertainty" in the GUM uncertainty framework) might be the outcome.

The GUM [ISO/IEC Guide 98-3:2008, 3.4.8] "…provides a framework for assessing uncertainty …", based on the law of propagation of uncertainty [ISO/IEC Guide 98-3:2008, Clause 5] and the characterization of the output quantity by a Gaussian distribution or a scaled and shifted t-distribution [ISO/IEC Guide 98-3:2008, G.6.2, G.6.4]. Within that framework, the law of propagation of uncertainty provides a means for propagating uncertainties through the model. Specifically, it evaluates the standard uncertainty associated with an estimate of the output quantity, given

1) best estimates of the input quantities,

2) the standard uncertainties associated with these estimates, and, where appropriate,

3) degrees of freedom associated with these standard uncertainties, and

4) any non-zero covariances associated with pairs of these estimates.

Also within the framework, the PDF taken to characterize the output quantity is used to provide a coverage interval, for a stipulated coverage probability, for that quantity.

The best estimates, standard uncertainties, covariances and degrees of freedom summarize the information available concerning the input quantities. With the approach considered here, the available information is encoded in terms of PDFs for the input quantities. The approach operates with these PDFs in order to determine the PDF for the output quantity.

Whereas there are some limitations to the GUM uncertainty framework, the propagation of distributions will always provide a PDF for the output quantity that is consistent with the PDFs for the input quantities. This PDF for the output quantity describes the knowledge of that quantity, based on the knowledge of the input quantities, as described by the PDFs assigned to them. Once the PDF for the output quantity is available, that quantity can be summarized by its expectation, taken as an estimate of the quantity, and its standard deviation, taken as the standard uncertainty associated with the estimate. Further, the PDF can be used to obtain a coverage interval, corresponding to a stipulated coverage probability, for the output quantity.

The use of PDFs as described in this Supplement is generally consistent with the concepts underlying the GUM. The PDF for a quantity expresses the state of knowledge about the quantity, i.e. it quantifies the degree of belief about the values that can be assigned to the quantity based on the available information. The information usually consists of raw statistical data, results of measurement, or other relevant scientific statements, as well as professional judgement.

In order to construct a PDF for a quantity, on the basis of a series of indications, Bayes' theorem can be applied [27, 33]. When appropriate information is available concerning systematic effects, the principle of maximum entropy can be used to assign a suitable PDF [51, 56].

The propagation of distributions has wider application than the GUM uncertainty framework. It works with richer information than that conveyed by best estimates and the associated standard uncertainties (and degrees of freedom and covariances when appropriate).

Decimal sign: The decimal sign in the English text is the point on the line, and the comma on the line is the decimal sign in the French text. (See 4.12)

An historical perspective is given in Annex A.

NOTE 1 The GUM provides an approach when linearization is inadequate [ISO/IEC Guide 98-3:2008, 5.1.2 Note]. The approach has limitations: only the leading non-linear terms in the Taylor series expansion of the model are used, and the PDFs for the input quantities are regarded as Gaussian.

NOTE 2 Strictly, the GUM characterizes the variable $(Y - y)/u(y)$ by a t-distribution, where Y is the output quantity, y an estimate of Y, and $u(y)$ the standard uncertainty associated with y [ISO/IEC Guide 98-3:2008, G.3.1]. This characterization is also used in this Supplement. [The GUM in fact refers to the variable $(y - Y)/u(y)$.]

NOTE 3 A PDF for a quantity is not to be understood as a frequency density.

NOTE 4 "The evaluation of uncertainty is neither a routine task nor a purely mathematical one; it depends on detailed knowledge of the nature of the measurand and of the measurement method and procedure used. The quality and utility of the uncertainty quoted for the result of a measurement therefore ultimately depends on the understanding, critical analysis, and integrity of those who contribute to the assignment of its value." [17].

0.2 JCGM background information

In 1997, the Joint Committee for Guides in Metrology (JCGM), chaired by the Director of the Bureau International des Poids et Mesures (BIPM), was created by the seven international organizations that had originally in 1993 prepared the *Guide to the expression of uncertainty in measurement (GUM)* and the *International vocabulary of basic and general terms in metrology (VIM)*. The JCGM assumed responsibility for these two documents from the ISO Technical Advisory Group 4 (TAG4).

The Joint Committee is formed by the BIPM with the International Electrotechnical Commission (IEC), the International Federation of Clinical Chemistry and Laboratory Medicine (IFCC), the International Organization for Standardization (ISO), the International Union of Pure and Applied Chemistry (IUPAC), the International Union of Pure and Applied Physics (IUPAP), and the International Organization of Legal Metrology (OIML). A further organization joined these seven international organizations, namely, the International Laboratory Accreditation Cooperation (ILAC).

JCGM has two Working Groups. Working Group 1, "Expression of uncertainty in measurement", has the task to promote the use of the GUM and to prepare Supplements and other documents for its broad application. Working Group 2, "Working Group on International vocabulary of basic and general terms in metrology (VIM)", has the task to revise and promote the use of the VIM. For further information on the activity of the JCGM, see www.bipm.org.

Supplements such as this one are intended to give added value to the GUM by providing guidance on aspects of uncertainty evaluation that are not explicitly treated in the GUM. The guidance will, however, be as consistent as possible with the general probabilistic basis of the GUM.

Uncertainty of measurement

Part 3:
Guide to the expression of uncertainty in measurement (GUM:1995)

Supplement 1:
Propagation of distributions using a Monte Carlo method

1 Scope

This Supplement provides a general numerical approach, consistent with the broad principles of the GUM [ISO/IEC Guide 98-3:2008, G.1.5], for carrying out the calculations required as part of an evaluation of measurement uncertainty. The approach applies to arbitrary models having a single output quantity where the input quantities are characterized by any specified PDFs [ISO/IEC Guide 98-3:2008, G.1.4, G.5.3].

As in the GUM, this Supplement is primarily concerned with the expression of uncertainty in the measurement of a well-defined physical quantity—the measurand—that can be characterized by an essentially unique value [ISO/IEC Guide 98-3:2008, 1.2].

This Supplement also provides guidance in situations where the conditions for the GUM uncertainty framework [ISO/IEC Guide 98-3:2008, G.6.6] are not fulfilled, or it is unclear whether they are fulfilled. It can be used when it is difficult to apply the GUM uncertainty framework, because of the complexity of the model, for example. Guidance is given in a form suitable for computer implementation.

This Supplement can be used to provide (a representation of) the PDF for the output quantity from which

a) an estimate of the output quantity,

b) the standard uncertainty associated with this estimate, and

c) a coverage interval for that quantity, corresponding to a specified coverage probability

can be obtained.

Given (i) the model relating the input quantities and the output quantity and (ii) the PDFs characterizing the input quantities, there is a unique PDF for the output quantity. Generally, the latter PDF cannot be determined analytically. Therefore, the objective of the approach described here is to determine a), b), and c) above to a prescribed numerical tolerance, without making unquantified approximations.

For a prescribed coverage probability, this Supplement can be used to provide any required coverage interval, including the probabilistically symmetric coverage interval and the shortest coverage interval.

This Supplement applies to input quantities that are independent, where each such quantity is assigned an appropriate PDF, or not independent, i.e. when some or all of these quantities are assigned a joint PDF.

Typical of the uncertainty evaluation problems to which this Supplement can be applied include those in which

— the contributory uncertainties are not of approximately the same magnitude [ISO/IEC Guide 98-3:2008, G.2.2],

— it is difficult or inconvenient to provide the partial derivatives of the model, as needed by the law of propagation of uncertainty [ISO/IEC Guide 98-3:2008, Clause 5],

— the PDF for the output quantity is not a Gaussian distribution or a scaled and shifted t-distribution [ISO/IEC Guide 98-3:2008, G.6.5],

— an estimate of the output quantity and the associated standard uncertainty are approximately of the same magnitude [ISO/IEC Guide 98-3:2008, G.2.1],

— the models are arbitrarily complicated [ISO/IEC Guide 98-3:2008, G.1.5], and

— the PDFs for the input quantities are asymmetric [ISO/IEC Guide 98-3:2008, G.5.3].

A validation procedure is provided to check whether the GUM uncertainty framework is applicable. The GUM uncertainty framework remains the primary approach to uncertainty evaluation in circumstances where it is demonstrably applicable.

It is usually sufficient to report measurement uncertainty to one or perhaps two significant decimal digits. Guidance is provided on carrying out the calculation to give reasonable assurance that in terms of the information provided the reported decimal digits are correct.

Detailed examples illustrate the guidance provided.

This document is a Supplement to the GUM and is to be used in conjunction with it. Other approaches generally consistent with the GUM may alternatively be used. The audience of this Supplement is that of the GUM.

NOTE 1 This Supplement does not consider models that do not define the output quantity uniquely (for example, involving the solution of a quadratic equation, without specifying which root is to be taken).

NOTE 2 This Supplement does not consider the case where a prior PDF for the output quantity is available, but the treatment here can be adapted to cover this case [16].

2 Normative references

The following referenced documents are indispensable for the application of this document. For dated references, only the edition cited applies. For undated references, the latest edition of the referenced document (including any amendments) applies.

ISO/IEC Guide 98-3:2008, *Uncertainty of measurement — Part 3: Guide to the expression of uncertainty in measurement (GUM:1995)*

ISO/IEC Guide 99:2007, *International vocabulary of metrology — Basic and general concepts and associated terms (VIM)*